MA1512 – DIFFERENTIAL EQUATIONS FOR ENGINEERING

A BOOK OF EXAMPLES

Contents

1
An Introduction to Differential Equations

1.1 First Principles

- A **differential equation** is an equation involving an (often unknown) function and its derivatives.
- An **ordinary differential equation** involves an independent variable (say, x), a function (say, $y(x)$), and one or more of its derivatives.
- The **order** of a differential equation is the order of the highest derivative that appears in the equation.
- A **solution** to a differential equation is a function that satisfies the equation.
- A **general solution** to a differential equation is a family of infinitely many possible solutions, often involving arbitrary constants. With additional information such as *initial conditions*, we can determine a **particular solution** that no longer involves arbitrary constants.
- A first-order differential equation is **separable** if it can be rewritten as

  $$M(x)\ dx = N(y)\ dy.$$

  To solve this, directly integrate both sides of the equation.
- In some cases, we'll need a substitution in order to make the equation separable.
  - If $y' = f(ax + by + c)$, we employ a *linear change of variable.* Let

    $$u = ax + by + c \quad \Longrightarrow \quad u' = a + by'.$$

  - If $y' = f(y/x)$, we let $y = xv$, and $y' = xv' + v$.

Example 1. The decomposition of a radioactive element is proportional to the amount of substance present at any given time. Model this phenomena by means of a differential equation, and find its solution.

Solution. Let $x(t)$ denote the amount of radioactive substance present at a given time t. The derivative dx/dt represents the rate at which the amount of substance changes over time, and we have

$$\frac{dx}{dt} \propto x.$$

As an equation, we introduce a proportionality constant—since radioactive substances decrease in amount over time, we know that this constant must be negative. We thus have the first-order differential equation

$$\frac{dx}{dt} = -kx, \text{ where } k > 0.$$

We check that the units agree with one another: x is measured in units of mass, dx/dt is measured in pass per unit time; thus, k must have units $[\text{time}]^{-1}$. The constant k is known as the **decay rate** of the element.

There is an 'obvious' solution: the constant *zero function* $x(t) = 0$ satisfies the differential equation. This, however, is not particularly interesting, as this would assume that there is no amount of radioactive substance at any given time. We may thus assume that $x(t) \neq 0$: in this case, we may divide throughout by x to yield

$$\frac{1}{x}\frac{dx}{dt} = -k.$$

By the chain rule of differentiation, the left-hand side is the derivative of $\ln|x|$ with respect to t. Thus,

$$\frac{d}{dt}\ln|x| = -k.$$

Integrating both sides with respect to t then yields

$$\ln|x| = -kt + c,$$

for some arbitrary constant $c \in \mathbb{R}$. Taking the exponential of both sides, we arrive at a solution:

$$x(t) = e^{-kt+c} = Ae^{-kt},$$

where $A = e^c$ is an arbitrary constant. Note that when $t = 0$, we have $x(0) = A$: our arbitrary constant A thus gains some physical meaning—it must represent the initial amount of substance present.

To determine the value of the decay constant k, more information, such as the *half-life* of the radioactive substance, is required. If the half-life is known to be τ, then it takes τ amount of time for the substance to decrease by half: $x(\tau) = x_0/2$. Plugging this into our solution yields

$$\frac{x_0}{2} = x_0 e^{-k\tau} \quad \Longrightarrow \quad k = \frac{\ln 2}{\tau}.$$

Note that, indeed, k must be positive, as we have established when we set up the differential equation.

Example 2. Solve the differential equation

$$\frac{dy}{dx} = e^x \left(1 + y^2\right).$$

Solution. We can solve this equation by a separation of variables:

$$\int \frac{1}{1+y^2}\,dy = \int e^x\,dx \quad \Longrightarrow \quad \tan^{-1} y = e^x + c.$$

We have a general solution, which we may rewrite as $y = \tan\left(e^x + c\right)$ (though this is not necessary).

Example 3. Solve the differential equation

$$\frac{dy}{dx} = \frac{1 - 2y - 4x}{1 + y + 2x}.$$

Solution. Observe that we may rewrite the differential equation as

$$\frac{dy}{dx} = \frac{1 - 2\left(y + 2x\right)}{1 + \left(y + 2x\right)}.$$

We employ a *linear change of variable*: let

$$u = y + 2x \quad \Longrightarrow \quad \frac{du}{dx} = \frac{dy}{dx} + 2.$$

Thus, our differential equation becomes

$$\frac{du}{dx} = \frac{1-2u}{1+u} + 2 \quad \Longrightarrow \quad \frac{du}{dx} = \frac{3}{1+u}.$$

This is now a separable equation!

$$\int 1 + u\,du = \int 3\,dx \quad \Longrightarrow \quad u + \frac{u^2}{2} = 3x + c.$$

Since $u = y + 2x$, we thus have the general solution

$$y + 2x + \frac{\left(y+2x\right)^2}{2} = 3x + c.$$

Example 4. Solve the differential equation

$$2xy\frac{dy}{dx} - y^2 + x^2 = 0.$$

Solution. Observe that we may rewrite the differential equation as

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = \frac{1}{2}\left(\frac{y}{x} - \frac{x}{y}\right).$$

Consider the substitution $u = y/x$, or $y = xu$, such that $y' = u + xu'$. Substituting this into the differential equation yields

$$u + x\frac{du}{dx} = \frac{1}{2}\left(u - \frac{1}{u}\right) \quad \Longrightarrow \quad x\frac{du}{dx} = -\frac{1}{2}\left(u + \frac{1}{u}\right).$$

Observe that this is now a separable differential equation!

$$\int \frac{2u}{u^2+1}\,du = \int -\frac{1}{x}\,dx \quad \Longrightarrow \quad \ln\left(u^2+1\right) = -\ln x + c.$$

Since $u = y/x$, we thus obtain the general solution

$$\ln\left(\frac{y^2}{x} + x\right) = c \quad \Longrightarrow \quad \frac{y^2}{x} + x = A.$$

1.2 The Geometry of Differential Equations

- A **direction field** (or a **slope field**) is a graph representing how the solution to a differential equation changes at various points. To sketch a direction field corresponding to a differential equation $y' = f(x, y)$, simply select various points (a, b) on the xy-plane; at each of these points, draw a short line segment whose slope is given by $f(a, b)$.
- An **equilibrium solution** of a differential equation is a solution that is constant; these correspond to horizontal lines on a direction field. An equilibrium solution $y(t) = \beta$ is said to be **stable** if solutions about this equilibrium approach β as $t \to \infty$. Otherwise, the equilibrium point is said to be **unstable**.

Example 5. Sketch a direction field for and determine any equilibrium points of the differential equation

$$y' = y.$$

Solution. If $y(x)$ is a solution to the differential equation, then the slope at every point (a, b) on the graph of $y(x)$ is simply b.

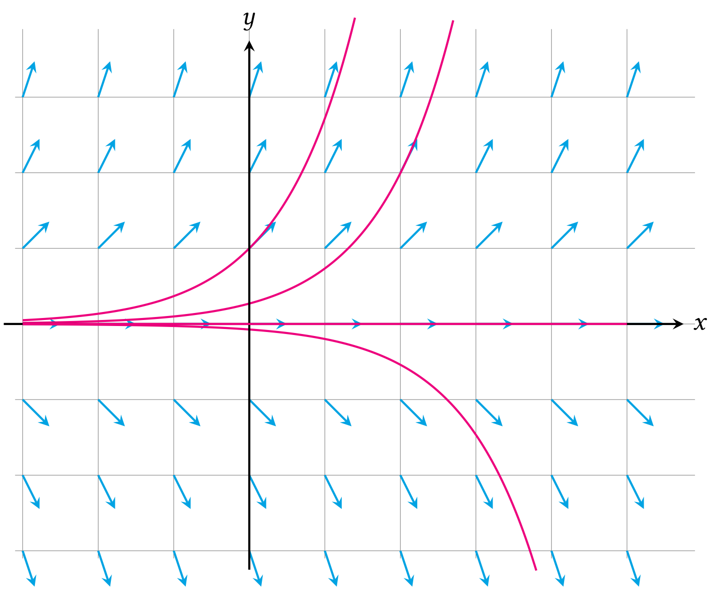


Figure 1: A slope field for $y' = y$. The graphs of particular solutions $y(x) = Ae^{x}$ for various values of A have been plot in red.

The slope field allows us to predict the long-term behavior of the solution $y(x)$ without finding it explicitly: in this example, we find that, depending on the initial conditions, the solution $y(x)$ can either infinitely increase, infinitely decrease, or stay constant at $y = 0$. In particular,

- the differential equation has only one unstable equilibrium, at $y = 0$;
- when $y(0) > 0$, then $y' > 0$ and any solution $y(x)$ increases towards positive infinity;
- when $y(0) < 0$, then $y' < 0$ and any solution $y(x)$ decreases towards negative infinity.

Example 6. Consider the following differential equation:

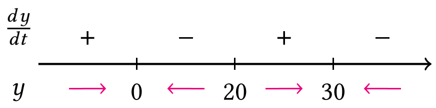
$$\frac{dy}{dt} = -10y\,(20-y)\left(1-\frac{1}{30}y\right).$$

1. Determine the equilibrium solutions of this differential equation, and state whether each equilibrium point is stable or unstable.
2. Determine the value of y as $t \longrightarrow \infty$ when $y\,(0) = 15$. Does your answer change when $y\,(0) = 150$?

Solution. The equilibrium solutions occur when $dy/dt = 0$: we thus have $y = 0, y = 20$, and $y = 30$. Observe that the sign of dy/dt changes as we plug in different values of y :

$$\frac{dy}{dt} = \begin{cases} + & \text{when } x < 0 \text{ and } 20 < x < 30, \\ - & \text{when } 0 < x < 20 \text{ and } x > 30. \end{cases}$$

We can sketch out a sign diagram for dy/dt.



Thus, if $y\,(0) = 15$, we find that $y' < 0$ and y decreases towards the equilibrium at 0 as $t \longrightarrow \infty$. When $y\,(0) = 150$, we find that $y' < 0$ and y decreases towards the equilibrium at 30 as t increases.

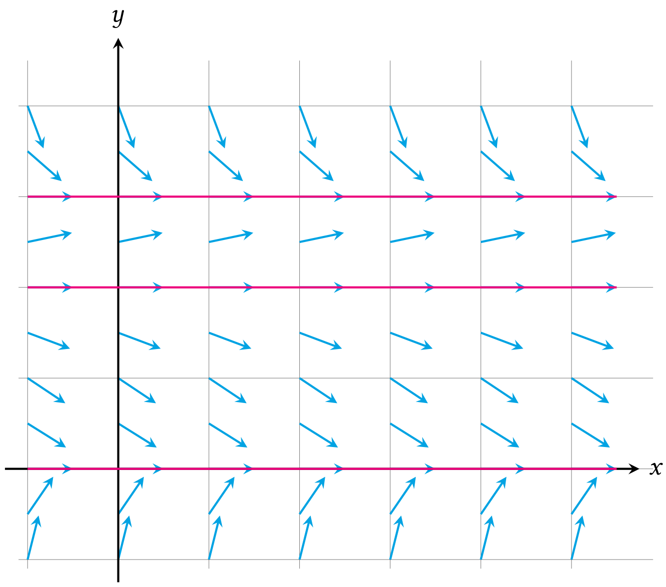


Figure 2: A sketch of the slope field for the given differential equation. The equilibrium solutions have been graphed in red.

1.3 Population Dynamics

- The **Malthusian model** assumes that the rate of change of a population is proportional to its present value:

$$\frac{dy}{dt} = ky \implies y(t) = y_0 e^{kt}, \text{ where } y_0 = y(0).$$

The Malthusian model assumes that, in the absence of limiting factors, a population would grow exponentially with *growth rate* k.

- The **Verhulst model** assumes that the growth rate varies according to the present value y of the population. When y is small, the growth rate is approximately constant; this growth rate decreases as the population y increases, eventually becoming negative upon exceeding the population's *carrying capacity* y_∞ :

$$\frac{dy}{dt} = \left[k\left(1 - \frac{y}{y_\infty}\right)\right] y.$$

The Verhulst model assumes that a population grows logistically, such that given any initial population,

$$\lim_{t\to\infty} y(t) = y_\infty.$$

Example 7. Solve the Verhulst differential equation given a known initial population:

$$\frac{dy}{dt} = \left[k\left(1 - \frac{y}{y_\infty}\right)\right] y, \quad y(0) = y_0.$$

Solution. We perform a separation of variables:

$$\int \frac{1}{y(1 - y/y_\infty)}\, dy = \int k\, dt.$$

To evaluate the integral on the left-hand side, we perform a partial fraction decomposition:

$$\frac{1}{y(1 - y/y_\infty)} = \frac{y_\infty}{y(y_\infty - y)} = \frac{1}{y} + \frac{1}{y_\infty - y}.$$

Hence, we find that

$$\int \frac{1}{y} + \frac{1}{y_\infty - y}\, dy = \int k\, dt \implies \ln|y| - \ln|y_\infty - y| = kt + c.$$

Combining terms and exponentiating yields

$$\frac{y_\infty}{y} - 1 = Ae^{-kt} \implies y(t) = \frac{y_\infty}{1 + Ae^{-kt}}.$$

When $t = 0$, we have $y = y_0$, so $A = y_\infty/y_0 - 1$. Thus, we find that

$$y(t) = \frac{y_\infty}{1 + (y_\infty/y_0 - 1)\, e^{-kt}}.$$

Example 8. A certain population behaves according to the Verhulst model with time measured in years. A constant number E of this species is captured per year. Model this situation as a differential equation. Determine any equilibrium solutions, and hence, find the largest possible number E of species that can be captured such that the population does not go extinct.

Solution. We may directly modify the Verhulst equation to account for the harvesting of species:

$$\frac{dy}{dt} = \left[k\left(1 - \frac{y}{y_\infty}\right)\right] y - E = -\frac{k}{y_\infty} y^2 + ky - E.$$

The equilibria of this differential equation can be computed with the quadratic formula:

$$y = \frac{-k \pm \sqrt{k^2 - 4kE/y_\infty}}{-2k/y_\infty} = \frac{k \mp \sqrt{k^2 - 4kE/y_\infty}}{2k/y_\infty}.$$

Note that if $k^2 - 4kE/y_\infty < 0$, then the differential equation has no equilibrium points. In particular, dy/dt will always be negative: any solution to the differential equation will be a decreasing function, and the population is bound to disappear. We thus require that

$$k^2 - 4kE/y_\infty \geq 0 \quad \Longrightarrow \quad E \leq ky_\infty/4,$$

and we can capture at most $E = ky_\infty/4$ of the species per year without causing the population to go extinct. We now have a few different possibilities for the equilibria of the differential equation:

Case 1. If $E = ky_\infty/4$, there is one equilibrium point at $\beta = y_\infty/2$. This is an *unstable* equilibrium: if $y_0 > \beta$, then the population decreases towards the equilibrium; however, if $y_0 < \beta$, the population decreases rapidly until extinction.

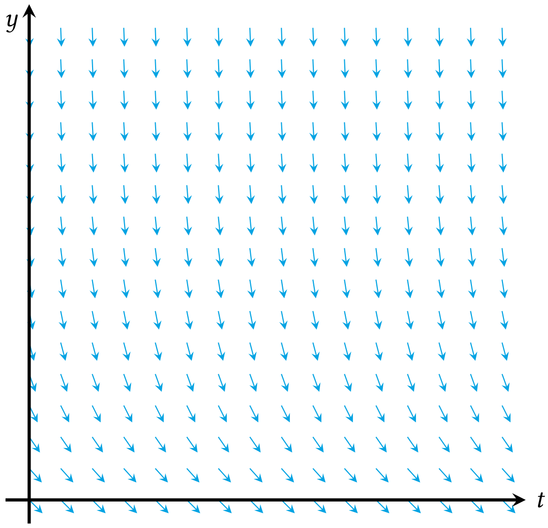
Case 2. If $E < ky_\infty/4$, then the differential equation has two equilibrium points:

$$\beta_1 = \frac{k - \sqrt{k^2 - 4kE/y_\infty}}{2k/y_\infty}, \quad \beta_2 = \frac{k + \sqrt{k^2 - 4kE/y_\infty}}{2k/y_\infty}.$$

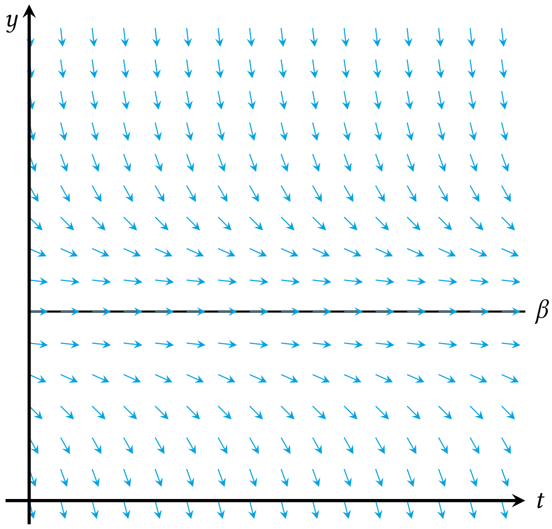
The equilibrium point at $y = \beta_2$ is *stable*, since small fluctuations about this point will allow the population to return to this equilibrium. On the other hand, $y = \beta_1$ is an *unstable* equilbrium since a fluctuation about this point can potentially lead the population to extinction.

We may visualize the populations' trajectories by roughly sketching out some slope fields of the differential equation for various values of E, based on the signs (i.e., positivity or negativity) of dy/dt.

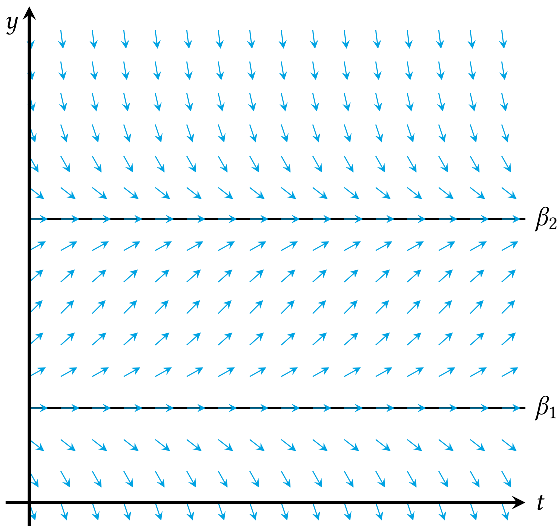
Figure 3: Sketches of slope fields of the modified Verhulst equation for various values of E.



When $E > ky_\infty/4$, the differential equation has no equilibria, and the population is always bound for extinction.



When $E = ky_\infty/4$, there is one unstable equilibrium point, and slight fluctuations in the population may lead to extinction.



When $E < ky_\infty/4$, there are two equilibria. At the stable equilibrium β_2, slight fluctuations in the population will allow the species to recover.

2
Linear Differential Equations

2.1 First-Order Linear Equations

► A **first-order linear** differential equation is an equation of the form

$$a(x)\,y' + b(x)\,y = c(x)\,,\ \text{with } a(x) \neq 0.$$

We can solve this by the method of **integrating factors**:

1. Rewrite the entire equation in standard linear form:

$$y' + p(x)\,y = q(x)\,.$$

2. Calculate the integrating factor $u = e^{\int p(x)\,dx}$.
3. Multiply both sides of the equation by u:

$$u\,(y' + py) = uq \quad \Longrightarrow \quad (uy)' = uq.$$

4. Integrate both sides of the equation.

► A **Bernoulli differential equation** is an equation which can be expressed in the form

$$y' + p(x)\,y = q(x)\,y^n \quad \equiv \quad y^{-n}y' + y^{1-n}p(x) = q(x)\,.$$

To solve this differential equation, consider the substitution $v = y^{1-n}$, so $v' = (1-n)\,y^{-n}y'$. Then, the Bernoulli equation is simply

$$v' + v\,(1-n)\,p(x) = q(x)\,(1-n)\,.$$

This is a first-order linear differential equation, which can be solved using the method of integrating factors.

Example 9. Find the general solution to the differential equation

$$xy' - y = x^3.$$

Solution. Observe that the differential equation is linear, but not separable—we thus solve it by using an integrating factor. First, we we rewrite it in standard linear form: dividing throughout by x, we have

$$y' - \frac{1}{x}y = x^2.$$

We calculate the integrating factor: observe that $p(x) = -1/x$, and so our integrating factor is

$$\int p(x)\ dx = -\ln x \quad \Longrightarrow \quad u = e^{-\ln x} = \frac{1}{x}.$$

Multiplying both sides of the equation by the integrating factor yields

$$\frac{1}{x}\left(y' - \frac{1}{x}y\right) = \frac{1}{x}\left(x^2\right) \quad \Longrightarrow \quad \left(\frac{1}{x}y\right)' = x.$$

Integrating both sides with respect to x then yields the general solution

$$\frac{1}{x}y = \frac{x^2}{2} + c.$$

Example 10. Solve the following differential equation:

$$xy' - y = -xy^2.$$

Solution. Observe that the differential equation is Bernoulli, since it can be written in the form

$$y' - \frac{1}{x}y = -y^2.$$

Dividing throughout by y^2, we have

$$y^{-2}y' - \frac{1}{x}y^{-2} = -1.$$

Thus, we let $v = y^{-1}$, and $v' = -y^{-2}\,y'$. Hence,

$$-v' - \frac{1}{x}v = -1 \quad \text{or} \quad v' + \frac{1}{x}v = 1.$$

This is a linear equation with integrating factor $e^{\int 1/x\, dx} = x$. Hence,

$$\frac{dv}{dx}x + v = x \quad \Longrightarrow \quad (xv)' = x \quad \Longrightarrow \quad xv = \frac{1}{2}x^2 + c.$$

Substituting the original value of v back, we end up with

$$\frac{x}{y} = \frac{1}{2}x^2 + c.$$

Example 11. An object of mass m, initially at rest is dropped from a certain height. As it falls towards the earth, it experiences a downward gravitational force, as well as an upward drag force. It is known that this force due to air resistance is directly proportional to the object's speed. Use Newton's Second Law to set up a differential equation modeling this system. Hence, or otherwise, determine the downward displacement x travelled by the object as a function of time t.

Solution. By Newton's Second Law, the sum of forces acting on an object is equal to the product of the object's mass and acceleration, $F = ma$, where $a(t)$ denotes the object's acceleration at time t. In this case, we have two forces acting on the object: we have

- a force due to gravity, $F_{\text{gravity}} = mg$, where $g \approx 9.81\ldots\text{m s}^{-2}$ is the constant acceleration due to gravity, and
- a drag force in the opposite direction: $F_{\text{drag}} = -kv$, where $v(t)$ denotes the velocity of the object at time t, and k is some constant.

Since acceleration is the first time derivative of velocity, we thus have the differential equation

$$mv' = mg - kv.$$

This is a linear differential equation: writing it in standard form, we have

$$v' + \frac{k}{m}v = g.$$

We have the integrating factor $e^{\int k/m\,dt} = e^{kt/m}$. Multiplying this to both sides of the equation yields

$$e^{kt/m}\left(v' + \frac{k}{m}v\right) = e^{kt/m}g \quad\Longrightarrow\quad \left(e^{kt/m}v\right)' = e^{kt/m}g.$$

Integrating both sides of the equation with respect to t yields

$$e^{kt/m}v = \frac{mg}{k}e^{kt/m} + c.$$

Since the object starts at rest, we know that $v(0) = 0$. Thus, $c = -mg/k$, ad we have the solution to the differential equation

$$v(t) = \frac{mg}{k} - \frac{mg}{k}e^{-kt/m} = \frac{mg}{k}\left(1 - e^{-kt/m}\right).$$

To determine the distance $x(t)$ travelled by the object, we can simply integrate our expression for the velocity:

$$x(t) = \int \frac{mg}{k}\left(1 - e^{-kt/m}\right)dt = \frac{mg}{k}\left(t + \frac{m}{k}e^{-kt/m}\right) + c.$$

Setting the starting point as the origin, we have $x(0) = 0$, and so $c = -m^2g/k$, and the displacement of the object is given by

$$x(t) = \frac{mg}{k}\left(t + \frac{m}{k}\left(e^{-kt/m} - 1\right)\right).$$

Remark. Note that the differential equation can also be solved by a separation of variables—try it out!

2.2 Higher-Order Linear Differential Equations

- A **linear differential equation with constant coefficients** is an equation of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x), \text{ where } a_i \in \mathbb{R}.$$

When $f(x) = 0$, the equation is said to be **homogeneous**; otherwise, the differential equation is **non-homogeneous**.

- Consider the *homogeneous* linear differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

To determine solutions to this differential equation, we make a guess: we try $y(x) = e^{\lambda x}$. Plugging this into the differential equation yields its **characteristic equation**:

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0.$$

  – If $\lambda \in \mathbb{R}$ is a real, distinct root, then a solution is given by

$$e^{\lambda x}.$$

  – If $\lambda \in \mathbb{R}$ is a repeated root with multiplicity r, then solutions are obtained by modifying our trial solution by a factor of x:

$$e^{\lambda x}, \ x e^{\lambda x}, \ x^2 e^{\lambda x}, \ \ldots, \ x^{r-1} e^{\lambda x}.$$

  – If $\lambda, \bar{\lambda} \in \mathbb{C}$ are conjugate roots $\alpha \pm i\beta$, then Euler's formula yields

$$e^{(\alpha \pm i\beta)x} = e^{\alpha x} \cos \beta x \pm i e^{\alpha x} \sin \beta x.$$

Extracting the real and imaginary parts yields the solutions

$$e^{\alpha x} \cos \beta x, \ e^{\alpha x} \sin \beta x.$$

If $\lambda, \bar{\lambda} \in \mathbb{C}$ are *repeated* roots, then we can extend the result on real repeated roots by modifying our solutions with a factor of x:

$$\begin{gathered} e^{\alpha x} \cos \beta x, \ x e^{\alpha x} \cos \beta x, \ x^2 e^{\alpha x} \cos \beta x, \ \ldots, \\ e^{\alpha x} \sin \beta x, \ x e^{\alpha x} \sin \beta x, \ x^2 e^{\alpha x} \sin \beta x, \ \ldots. \end{gathered}$$

Theorem 12 (Superposition Principle). *Let $y_1(x)$ and $y_2(x)$ be solutions to a homogeneous linear differential equation*

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Then, a solution to the differential equation is also given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \text{ for all } c_1, c_2 \in \mathbb{R}.$$

Example 13. Find the general solution to

$$4y'' + 12y' + 9y = 0.$$

Solution. The characteristic equation is

$$4\lambda^2 + 12\lambda + 9 = (2\lambda + 3)^2 = 0,$$

with a repeated root $\lambda = -3/2$. Thus, solutions are given by $e^{-\frac{3}{2}x}$ and $xe^{-\frac{3}{2}x}$, and the general solution is given by

$$y = c_1 e^{-\frac{3}{2}x} + c_2 x e^{-\frac{3}{2}x}.$$

Example 14. Find the general solution to

$$y'' - 6y' + 13y = 0.$$

Solution. The characteristic equation is

$$\lambda^2 - 6\lambda + 13 = 0.$$

By the quadratic formula, we find complex conjugate roots:

$$\lambda = \frac{6 \pm \sqrt{36 - 52}}{2} = 3 \pm 2i.$$

Thus, solutions are given by $e^{3x}\cos 2x$ and $e^{3x}\sin 2x$, and we may form the general solution

$$y = c_1 e^{3x}\cos 2x + c_2 e^{3x}\sin 2x.$$

Example 15. Find the particular solution to the following differential equation:

$$y'' - y = 0, \quad y(0) = 5, \quad y'(0) = 3.$$

Solution. The characteristic equation is

$$\lambda^2 - 1 = 0,$$

which has distinct roots $\lambda = -1, 1$. We thus have the general solution

$$y(x) = c_1 e^{-x} + c_2 e^{x}.$$

To determine the values of c_1 and c_2, we plug in the initial conditions. Note that

$$y'(x) = -c_1 e^{-x} + c_2 e^{x}.$$

Thus, when $x = 0$,

$$y(0) = c_1 + c_2 = 5, \quad y'(0) = -c_1 + c_2 = 3.$$

We can now solve for the values of $c_1$and c_2:

$$\begin{cases} c_1 & + & c_2 & = & 5 \\ -c_1 & + & c_2 & = & 3 \end{cases} \implies \begin{cases} c_1 & + & c_2 & = & 5 \\ & & 2c_2 & = & 8 \end{cases}.$$

Hence, $c_2 = 4$ and $c_1 = 1$, and the particular solution is thus

$$y(x) = e^{-x} + 4e^{x}.$$

Example 16. Determine the general solution of the following *fourth*-order differential equation

$$y^{(4)} - 5y'' - 36y = 0.$$

Solution. The characteristic equation is

$$\lambda^4 - 5\lambda^2 - 36 = \left(\lambda^2 - 9\right)\left(\lambda^2 + 4\right) = 0.$$

It has four roots:

- two real roots $\lambda = -3, 3$, from which we have solutions e^{-3x} and e^{3x};
- complex conjugate roots $\lambda = \pm 2i$, yielding $\cos 2x$ and $\sin 2x$.

We can thus construct the general solution:

$$y = c_1 e^{-3x} + c_2 e^{3x} + c_3 \cos 2x + c_4 \sin 2x.$$

3 The Harmonic Oscillator

3.1 Non-Homogeneous Linear Differential Equations

- Consider the *non-homogeneous* second-order linear equation

$$y'' + py' + qy = f(x), \quad \text{where} \quad f(x) \neq 0.$$

  The general solution to this differential equation is given by

$$y(x) = y_h(x) + y_p(x),$$

  where $y_h(x)$ is the general solution to the *complementary homogeneous equation* $y'' + py' + qy = 0$, and $y_p(x)$ is any particular solution.

- When $f(x)$ involves simple functions, we can attempt to 'guess' $y_p(x)$ by the **method of undetermined coefficients**. For example,

  - if $f(x)$ is a polynomial of degree n, we let y_p be an arbitrary nth degree polynomial with undetermined coefficients;
  - if $f(x)$ contains an exponential e^{kx}, then we try $y_p = Ae^{kx}$, where $A \in \mathbb{R}$ is an undetermined coefficient;
  - if $f(x)$ contains either $\cos kx = \mathfrak{Re}(e^{ikx})$ or $\sin kx = \mathfrak{Im}(e^{ikx})$, we can instead solve for the real or imaginary solutions, respectively, of the complex equation $y'' + my' + ny = e^{ikx}$ by trying $y_p = Ae^{ikx}$, where $A \in \mathbb{C}$ is an undetermined coefficient.

  Modification: If any term of the trial solution is already a solution of the complementary equation, multiply your trial solution by x.

- Given a solution $y_h(x)$ to the complementary equation, we can perform a **variation of parameters** to obtain a particular solution:

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) \quad \rightsquigarrow \quad y_p(x) = u(x)\, y_1(x) + v(x)\, y_2(x),$$

  where the functions $u(x)$ and $v(x)$ are given by

$$u(x) = -\int \frac{y_2 f}{y_1 y_2' - y_1' y_2}\, dx, \qquad v(x) = \int \frac{y_1 f}{y_1 y_2' - y_1' y_2}\, dx.$$

Example 17. Find the general solution to

$$y'' + 4y = 2e^{2x}.$$

Solution. The characteristic equation of the complementary equation is $\lambda^2 + 4 = 0$, with complex roots $\lambda = \pm 2i$, and so

$$y_h = c_1 \cos 2x + c_2 \sin 2x.$$

We now find a particular solution: since $f(x) = e^{2x}$, we might guess

$$y_p = Ae^{2x}.$$

Then, $y_p' = 2Ae^{2x}$ and $y_p'' = 4Ae^{2x}$, and substituting these into the equation,

$$4Ae^{2x} + 4\left(Ae^{2x}\right) = 2e^{2x} \quad \Longrightarrow \quad 8A = 2$$

so $A = 1/4$. Hence, we have the particular solution $y_p = e^{2x}/4$, and we obtainthe general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{e^{2x}}{4}.$$

Example 18. Find the general solution to

$$y'' + y' - 2y = 4x^2.$$

Solution. We begin by solving complementary equation $y'' + y' - 2y = 0$. The characteristic equation is

$$\lambda^2 + \lambda - 2\lambda = (\lambda - 1)(\lambda + 2) = 0 \quad \Longrightarrow \quad \lambda = 1, -2.$$

Hence, we have the solution of the complementary equation

$$y_h = c_1 e^x + c_2 e^{-2x}.$$

We now find a particular solution: observe that $f(x) = x^2$ is a polynomial of degree 2, so, we might guess a particular solution to be a polynomial of degree 2, as well: such a polynomial would take on the form

$$y_p = Ax^2 + Bx + C.$$

We now determine the coefficients A, B, and C. Observe that $y_p' = 2Ax + B$, and $y_p'' = 2A$. Substituting this into the differential equation,

$$\begin{aligned} 2A + 2Ax + B - 2\left(Ax^2 + Bx + C\right) &= 4x^2 \\ -2Ax^2 + (2A - 2B)x + (2A + B - 2C) &= 4x^2. \end{aligned}$$

Comparing coefficients, we find that

$$-2A = 4, \quad 2A - 2B = 0, \quad 2A + B - 2C = 0.$$

Solving this, we find that $A = -2$, $B = -2$, and $C = -3$. So,

$$y_p = -2x^2 - 2x - 3,$$

and hence, the general solution is

$$y = y_h + y_p = c_1 e^x + c_2 e^{-2x} - 2x^2 - 2x - 3.$$

Example 19. Solve

$$y'' + y' - 2y = \sin x.$$

Solution. From the previous example, we have

$$y_h = c_1 e^x + c_2 e^{-2x}.$$

We now find a particular solution: note that both cosine and sine terms can be differentiated to yield $\sin x$. In order to minimize dealing with sines and cosines, we recall Euler's identity:

$$e^{ix} = \cos x + i \sin x.$$

In particular, $\sin x = \mathfrak{Im}\left(e^{ix}\right)$. Hence, we can equivalently solve for the particular solution of the differential equation

$$y'' + y' - 2y = e^{ix},$$

noting that we only want the imaginary part of the solution.

Since $f(x) = e^{ix}$, we guess

$$y_p = Ae^{ix}.$$

Since i is simply a constant, the derivatives are given by

$$y_p' = Aie^{ix}, \quad y_p'' = -Ae^{ix}.$$

Plugging this into the differential equation yields

$$\underbrace{-Ae^{ix}}_{y''} + \underbrace{Aie^{ix}}_{y'} - \underbrace{2Aei^{ix}}_{2y} = e^{ix} \quad \Longrightarrow \quad A(-3+i) = 1.$$

Thus, we find that

$$A = \frac{1}{-3+i} = \frac{1}{-3+i} \cdot \frac{-3-i}{-3-i} = -\frac{3}{10} - \frac{1}{10}i.$$

Note that we only require the imaginary part of y_p :

$$\begin{aligned} y_p &= \left(-\frac{3}{10} - \frac{1}{10}i\right)(\cos x + i \sin x) \\ &= -\frac{3}{10}\cos x + \frac{1}{10}\sin x + i\left(-\frac{1}{10}\cos x - \frac{3}{10}\sin x\right). \end{aligned}$$

Thus, a particular solution is given by

$$\mathfrak{Im}\left(y_p\right) = -\frac{1}{10}\cos x - \frac{3}{10}\sin x.$$

Therefore, our general solution is

$$y = c_1 e^x + c_2 e^{-2x} - \frac{1}{10}\cos x - \frac{3}{10}\sin x.$$

Example 20. Find the general solution to

$$y'' - 3y' - 4y = 5e^{4x}.$$

Solution. The characteristic equation of the complementary equation is

$$\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0.$$

We thus have the general solution $y_h = c_1 e^{4x} + c_2 e^{-x}$.

We now find a particular solution: since $f(x) = e^{4x}$, we might guess $y_p = Ae^{4x}$. Note, however, that such a guess will not work (try plugging it into the equation!): in particular, if we were to construct the general solution $y = y_h + y_p$, this term 'disappears' into y_h, leaving us with no particular solution. We thus *modify* our guess: we instead try

$$\begin{aligned} y_p &= Axe^{4x}, \\ y_p' &= Ae^{4x} + 4Axe^{4x} = Ae^{4x}(1 + 4x), \\ y_p'' &= 4Ae^{4x}(1 + 4x) + 4Ae^{4x} = 4Ae^{4x}(2 + 4x). \end{aligned}$$

Substituting these into the original differential equation yields

$$\begin{aligned} 4Ae^{4x}(2 + 4x) - 3\left(Ae^{4x}(1 + 4x)\right) - 4\left(Axe^{4x}\right) &= 5e^{4x} \\ 8A + 16Ax - 3A - 12Ax - 4Ax &= 5. \end{aligned}$$

We find that $5A = 5$, and so our 'new' guess works with $A = 1$. We thus have $y_p = xe^{4x}$, and we have the general solution

$$y = c_1 e^{4x} + c_2 e^{-x} + xe^{4x}.$$

Example 21. Find a particular solution to

$$y'' + 2y' + 3y = 34e^x \cos 2x.$$

Solution. First, we observe that $f(x) = 34e^x \cos 2x$ is equal to

$$\mathfrak{Re}\left(34e^x e^{i2x}\right) = \mathfrak{Re}\left(34e^{x(1+2i)}\right).$$

Thus, we can equivalently solve for

$$y'' + 2y' + 3y = 34e^{x(1+2i)},$$

noting that we only want the real part of this solution. We try

$$\begin{aligned} y_p &= Ae^{x(1+2i)}, \\ y_p' &= A(1 + 2i)e^{x(1+2i)}, \\ y_p'' &= A(1 + 2i)^2 e^{x(1+2i)} = A(-3 + 4i)e^{x(1+2i)}. \end{aligned}$$

Substituting this into the equation, we have

$$A(-3 + 4i)e^{x(1+2i)} + 2A(1 + 2i)e^{x(1+2i)} + 3Ae^{x(1+2i)} = 34e^{x(1+2i)},$$

and $A(2 + 8i) = 34$. Simplifying this, we find

$$A = \frac{34}{2 + 8i} = \frac{17}{1 + 4i} \cdot \frac{1 - 4i}{1 - 4i} = \frac{17 - 68i}{17} = 1 - 4i.$$

So,

$$\begin{aligned} y_p &= (1 - 4i)e^{x(1+2i)} = e^x(1 - 4i)(\cos 2x + i \sin 2x) \\ &= e^x(\cos 2x + 4 \sin 2x + i(-4 \cos 2x + \sin 2x)). \end{aligned}$$

Thus, a particular solution is given by

$$\mathfrak{Re}(y_p) = e^x(\cos 2x + 4 \sin 2x).$$

Example 22. Find the general solution to

$$y'' + y = \tan x.$$

Solution. The characteristic equation of the complementary equation is $\lambda^2 + 1 = 0$, with complex roots $\lambda = \pm i$. Thus, we have

$$y_h = c_1 \cos x + c_2 \sin x.$$

We can produce a particular solution by varying the parameters: letting $f(x) = \tan x$, $y_1(x) = \cos x$ and $y_2(x) = \sin x$, we have

$$y_p = u(x) \cos x + v(x) \sin x,$$

with

$$u(x) = -\int \frac{y_2 f}{y_1 y_2' - y_1' y_2}\, dx, \qquad v(x) = \int \frac{y_1 f}{y_1 y_2' - y_1' y_2}\, dx.$$

Since $y_1' = -\sin x$ and $y_2' = \cos x$, we have

$$y_1 y_2' - y_1' y_2 = \cos^2 x - \left(-\sin^2 x\right) = 1.$$

Thus, we find that

$$\begin{aligned} u(x) &= -\int \sin x \tan x\, dx = -\int \frac{\sin^2 x}{\cos x}\, dx \\ &= -\int \frac{1 - \cos^2 x}{\cos x}\, dx = \int \cos x - \sec x\, dx \\ &= \sin x - \ln\left|\sec x + \tan x\right|, \\ v(x) &= \int \cos x \tan x\, dx = \int \sin x\, dx = -\cos x. \end{aligned}$$

Hence, a particular solution is given by

$$\begin{aligned} y_p &= \left[\sin x - \ln\left|\sec x + \tan x\right|\right] \cos x - \cos x \sin x \\ &= -\ln\left|\sec x + \tan x\right| \cos x. \end{aligned}$$

We thus have the general solution

$$y = c_1 \cos x + c_2 \sin x - \ln\left|\sec x + \tan x\right| \cos x.$$

3.2 Simple Harmonic Motion

- A particle is said to be in **simple harmonic motion** if its acceleration is directly proportional to its displacement:

$$\ddot{x} = -\omega^2 x \quad \equiv \quad \ddot{x} + \omega x = 0.$$

The constant ω denotes the **angular frequency** of the oscillation.

- Consider a **damped oscillator**, in which a resisting force proportional to velocity acts against an oscillating object:

$$\ddot{x} = -\omega^2 x - 2\gamma\dot{x} \quad \equiv \quad \ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0.$$

The characteristic equation is given by $\lambda^2 + 2\gamma\lambda + \omega^2 = 0$, with solutions

$$\lambda = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega^2}.$$

The nature of the object's motion now depends on the value of the discriminant $\Delta = \gamma^2 - \omega^2$:

- if $\omega^2 < \gamma^2$, we have real roots $\lambda_1, \lambda_2 \in \mathbb{R}$ and

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},$$

which describes the motion of an overdamped oscillator,

- if $\omega^2 = \gamma^2$, we have a repeated real root $\lambda \in \mathbb{R}$:

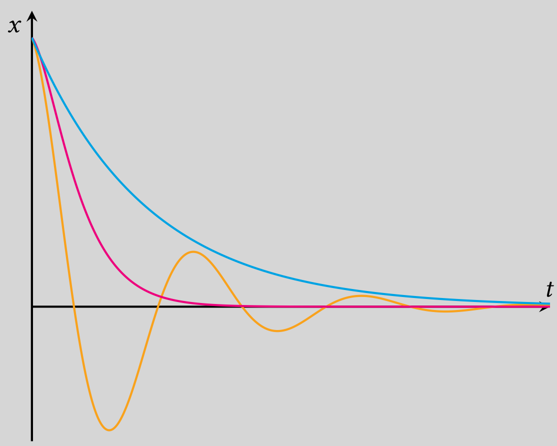
$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t},$$

and we have a critically damped oscillator,

- if $\omega^2 > \gamma^2$, we have complex conjugate roots $\lambda = \alpha + i\beta \in \mathbb{C}$, and

$$x(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t,$$

describing the motion of an underdamped oscillator.



Example 23. Find the general solution to the equation of motion of a simple harmonic oscillator:

$$\ddot{x} = -\omega^2 x.$$

Solution. We have a second-order homogeneous linear differential equation, whose characteristic equation is given by

$$\lambda^2 + \omega^2 = 0 \quad \Longrightarrow \quad \lambda = \pm i\omega.$$

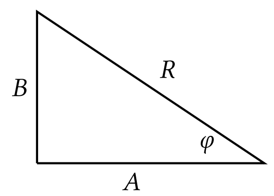
A general solution is thus given by

$$x(t) = A\cos\omega t + B\sin\omega t, \quad A, B \in \mathbb{R}.$$

Equivalently, we may write

$$x(t) = R\cos(\omega t - \varphi), \text{ where } R = \sqrt{A^2 + B^2}.$$

To see this, observe that A, B, and $R = \sqrt{A^2 + B^2}$ form the sides of a right triangle, as illustrated below. Let φ be the angle between the side of length A and the hypotenuse.



Then, $\cos\varphi = A/R$ and $\sin\varphi = B/R$, and we may rewrite our solution as:

$$\begin{aligned} x(t) &= A\cos\omega t + B\sin\omega t \\ &= R\left(\frac{A}{R}\cos\omega t + \frac{B}{R}\sin\omega t\right) \\ &= R(\cos\varphi\cos\omega t + \sin\varphi\sin\omega t) \\ &= R\cos(\omega t - \varphi). \end{aligned}$$

Example 24. Consider an undamped harmonic oscillator equipped with an external sinusoidal driving force, such as one supplied by a motor attached to the oscillator. The equation of motion is given by

$$\ddot{x} = -\omega^2 x + F\cos\psi t.$$

The phenomena of **resonance** occurs when the angular frequency of the driving force is equal to that of the oscillator: that is, $\psi = \omega$. Determine the function $x(t)$ that describes the oscillator's motion under resonance.

Solution. We wish to solve the non-homogeneous differential equation

$$\ddot{x} + \omega^2 x = F\cos\omega t.$$

We can do this using the method of undetermined coefficients. To begin, we already have the solution to the complementary equation

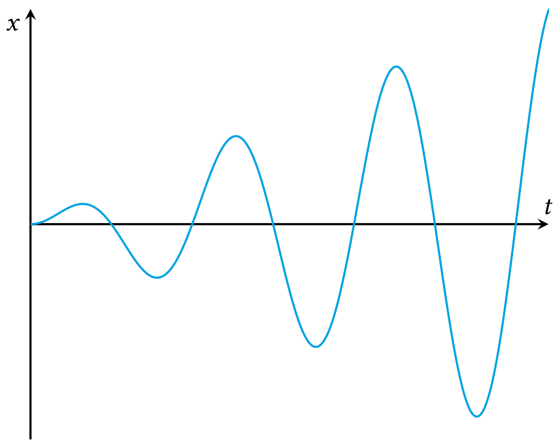
$$x_h = A\cos\omega t + B\sin\omega t = R\cos(\omega t - \varphi).$$

To find x_p, we consider the non-homogeneous part $F\cos\omega t = \mathfrak{Re}\left\{Fe^{i\omega t}\right\}$. We can thus consider the complex differential equation

$$\ddot{x} + \omega^2 x = Fe^{i\omega t},$$

noting that we only require the real part of this solution.

We can first try
$$x_p = Ce^{i\omega t}.$$
Note, however, that $\mathfrak{Re}\left\{Ce^{i\omega t}\right\} = C\cos\omega t$ appears in x_h; we thus need to modify our guess. Let
$$x_p = Cte^{i\omega t}.$$
Then,
$$\dot{x}_p = Ce^{i\omega t} + Ci\omega te^{i\omega t} = Ce^{i\omega t}\left(1 + i\omega t\right),$$
$$\ddot{x}_p = Ci\omega e^{i\omega t}\left(1 + i\omega t\right) + Ci\omega e^{i\omega t} = Ci\omega e^{i\omega t}\left(2 + i\omega t\right).$$
Plugging these into the differential equation yields:
$$Ci\omega e^{i\omega t}\left(2 + i\omega t\right) + \omega^2\left(Cte^{i\omega t}\right) = Fe^{i\omega t}$$
$$Ci\omega\left(2 + i\omega t\right) + Ct\omega^2 = F.$$
We thus find that $2Ci\omega = F$, or
$$C = \frac{F}{2i\omega}\cdot\frac{i}{i} = -i\frac{F}{2\omega}.$$
Our particular solution is thus
$$x_p = \left(-i\frac{F}{2\omega}\right)te^{i\omega t} = -i\frac{F}{2\omega}t\left(\cos\omega t + i\sin\omega t\right).$$
We only require the real part of this solution:
$$\mathfrak{Re}\left\{x_p\right\} = \frac{F}{2\omega}t\sin\omega t.$$
Thus, the motion of an oscillator under resonance is given by
$$x\left(t\right) = R\cos\left(\omega t - \varphi\right) + \frac{F}{2\omega}t\sin\omega t.$$
Observe that as t goes on, $x\left(t\right)$ continuously increases, and the amplitude of the oscillation grows until it goes out of control.



4
The Laplace Transform

4.1 Basic Properties

▸ The **Laplace transform of $f(t)$** is defined by

$$\mathcal{L}[f(t)] = F(s) = \lim_{h\to\infty}\int_0^h f(t)\cdot e^{-st}\,dt,$$

and $f(t)$ is the **inverse** Laplace transform of $F(s)$: $f(t) = \mathcal{L}^{-1}[F(s)]$.

Theorem 25 (Linearity). *Given functions $f(t)$ and $g(t)$,*

$$\mathcal{L}[af(t)+bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)], \text{ for all } a,b\in\mathbb{R}.$$

Theorem 26 (The First Shifting Theorem). *Let $F(s) = \mathcal{L}[f(t)]$. Then,*

$$\mathcal{L}\left[e^{at}f(t)\right] = F(s-a).$$

Theorem 27 (Derivatives). *Given a function $f(t)$,*

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Theorem 28 (Differentiation Property). *Let $F(s) = \mathcal{L}[f(t)]$. Then,*

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s).$$

Example 29. Use the definition of the Laplace transform to compute $\mathcal{L}[f(t)]$ for the function $f(t) = t$, $t \geq 0$.

Solution. We directly apply the definition of the Laplace transform:

$$\begin{aligned}\mathcal{L}[t] &= \lim_{h\to\infty}\int_0^h t\cdot e^{-st}\,dt\\ &= \lim_{h\to\infty}\left[e^{-st}\left(-\frac{t}{s}-\frac{1}{s^2}\right)\right]\Bigg|_0^h\\ &= \lim_{h\to\infty}\left(-\frac{he^{-sh}}{s}-\frac{e^{-sh}}{s^2}+\frac{1}{s^2}\right).\end{aligned}$$

This limit exists when $s > 0$:

$$\mathcal{L}[t] = \frac{1}{s^2}, \quad s > 0.$$

Example 30. Find the inverse Laplace transform of $F(s) = \dfrac{1}{s^2+2s-3}$.

Solution. We perform a partial fraction decomposition of $F(s)$:

$$F(s) = \frac{1}{s^2+2s-3} = -\frac{1}{4}\cdot\frac{1}{s+3} + \frac{1}{4}\cdot\frac{1}{s-1}.$$

We take the inverse transform of $F(s)$ term by term, with reference to the table of Laplace transforms:

$$\mathcal{L}^{-1}[F(s)] = -\frac{1}{4}\cdot\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] + \frac{1}{4}\cdot\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] = -\frac{1}{4}e^{-3t} + \frac{1}{4}e^{t}.$$

Example 31. Evaluate $\mathcal{L}\left[e^{at}\cos\omega t\right]$.

Solution. Let $F(s)$ be the Laplace transform of $\cos\omega t$. By the First Shifting Theorem, we simply replace every instance of s in $F(s)$ with $s-a$:

$$\mathcal{L}\left[e^{at}\cos\omega t\right] = F(s-a) = \frac{s-a}{(s-a)^2+\omega^2}.$$

Example 32. Find the inverse Laplace transform of

$$F(s) = \frac{1-3s}{s^2+8s+32}.$$

Solution. Observe that the denominator cannot be factored; thus, we try a different way of manipulating this expression. Completing the square,

$$s^2+8s+32 = \left(s^2+8s+4^2\right)+32-4^2 = (s+4)^2+16.$$

We may thus apply the First Shifting Theorem, but in order to do so, we need to rewrite the numerator.

$$\begin{aligned} F(s) &= \frac{1-3(s+4-4)}{(s+4)^2+16} = \frac{-3(s+4)+13}{(s+4)^2+16} \\ &= -\frac{3(s+4)}{(s+4)^2+4^2} + \frac{13}{(s+4)^2+4^2}. \end{aligned}$$

We now take the inverse Laplace transform of $F(s)$:

$$\begin{aligned} \mathcal{L}^{-1}[F(s)] &= -3\mathcal{L}^{-1}\left[\frac{s+4}{(s+4)^2+4^2}\right] + \frac{13}{4}\mathcal{L}^{-1}\left[\frac{4}{(s+4)^2+4^2}\right] \\ &= -3e^{-4t}\cos 4t + \frac{13}{4}e^{-4t}\sin 4t. \end{aligned}$$

Example 33. Use the definition of the Laplace transform to compute the Laplace transform of a given function's derivative, $f'(t)$.

Solution. We directly apply the definition of the Laplace transform:

$$\mathcal{L}[f'(t)] = \lim_{h\to\infty}\int_0^h e^{-st}f'(t)\,dt.$$

Integrating by parts, let $u = e^{-st}$ and $dv = f'(t)\,dt$. Then, $du = -se^{-st}dt$, $v = f(t)$, and

$$\begin{aligned} \mathcal{L}[f'(t)] &= \lim_{h\to\infty} e^{-st}f(t)\Big|_0^h - \int_0^h \left(-se^{-st}\right)f(t)\,dt \\ &= -f(0) + s\lim_{h\to\infty}\int_0^h e^{-st}f(t) \\ &= -f(0) + s\mathcal{L}[f(t)]. \end{aligned}$$

Example 34. Use the definition of the Laplace transform to show that, given a function $f(t)$,

$$\mathcal{L}[tf(t)] = -F'(s).$$

Solution. Let $F(s) = \mathcal{L}[f(t)]$. Differentiating this expression with respect to s, we have

$$\begin{aligned} F'(s) &= \frac{d}{ds} \lim_{h\to\infty} \int_0^h e^{-st} f(t)\, dt \\ &= \lim_{h\to\infty} \int_0^h (-t)\, e^{-st} f(t)\, dt \\ &= -\lim_{h\to\infty} \int_0^h e^{-st} [tf(t)]\, dt \\ &= -\mathcal{L}[tf(t)]. \end{aligned}$$

Example 35. Solve the initial value problem

$$y' + 2y = 0, \quad y(0) = 2.$$

Solution. Taking the Laplace transform of both sides of the differential equation, we have

$$\mathcal{L}[y' + 2y] = \mathcal{L}[0].$$

Evaluating the left-hand side, observe that

$$\begin{aligned} \mathcal{L}[y' + 2y] &= (s\mathcal{L}[y] - y(0)) + 2\mathcal{L}[y] \\ &= (s\mathcal{L}[y] - 2) + 2\mathcal{L}[y] \\ &= (s+2)\,\mathcal{L}[y] - 2. \end{aligned}$$

The right-hand side is simply $\mathcal{L}[0] = 0$. Thus, we have

$$(s+2)\,\mathcal{L}[y] - 2 = 0 \quad \Longrightarrow \quad \mathcal{L}[y] = \frac{2}{s+2}.$$

Therefore, taking the inverse Laplace transform of both sides, we arrive at the particular solution to the differential equation

$$y = 2 \cdot \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = 2e^{-2t}.$$

This is indeed correct—notice that the differential equation is separable, and we arrive at the same answer! Observe that

$$\frac{dy}{dt} + 2y = 0 \quad \Longrightarrow \quad \int \frac{1}{y}\, dy = \int -2\, dt \quad \Longrightarrow \quad \ln|y| = -2t + c.$$

Exponentiating both sides yields

$$y = e^{-2x+c} = Ae^{-2t},$$

and since $y(0) = 2$, we have $A = 2$, yielding the same particular solution

$$y = 2e^{-2t}.$$

Example 36. Solve the initial value problem

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution. We take the Laplace transform of both sides of the equation:

$$\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0].$$

We expand the left-hand side of the equation:

$$\begin{aligned}
\mathcal{L}[y'' - y' - 2y] &= \mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] \\
&= \left(s^2\mathcal{L}[y] - sy(0) - y'(0)\right) - (s\mathcal{L}[y] - y(0)) - 2\mathcal{L}[y] \\
&= \left(s^2\mathcal{L}(y) - 1\right) - (s\mathcal{L}[y] - 1) - 2\mathcal{L}[y] \\
&= \left(s^2 - s - 2\right)\mathcal{L}[y] - s + 1.
\end{aligned}$$

Since $\mathcal{L}[0] = 0$, we have

$$\left(s^2 - s - 2\right)\mathcal{L}[y] - s + 1 = 0 \quad \Longrightarrow \quad \mathcal{L}[y] = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}.$$

To compute the inverse Laplace transform of $\mathcal{L}[y]$, we take its partial fraction decomposition:

$$\frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} = \frac{(A+B)s + A - 2B}{s^2 - s - 2}.$$

Thus, $A + B = 1$ and $A - 2B = -1$. Hence, $B = 2/3$ and $A = 1/3$, and

$$\mathcal{L}[y] = \frac{s-1}{(s-2)(s+1)} = \frac{1}{3} \cdot \frac{1}{s-2} + \frac{2}{3} \cdot \frac{1}{s+1}.$$

Taking the inverse Laplace transform of both sides, we find that

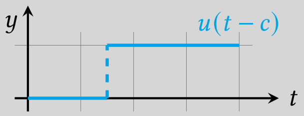
$$y = \mathcal{L}^{-1}\left[\frac{1}{3} \cdot \frac{1}{s-2} + \frac{2}{3} \cdot \frac{1}{s+1}\right] = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

Verify that you obtain the same result by solving the equation as a homogeneous linear equation!

4.2 *Step Functions and the Unit Impulse*

- The **unit step function**, denoted by $u(t-c)$, is defined by

$$u(t-c) = \begin{cases} 0, & t < c, \\ 1, & t > c. \end{cases}$$



The Laplace transform of the unit step is given by

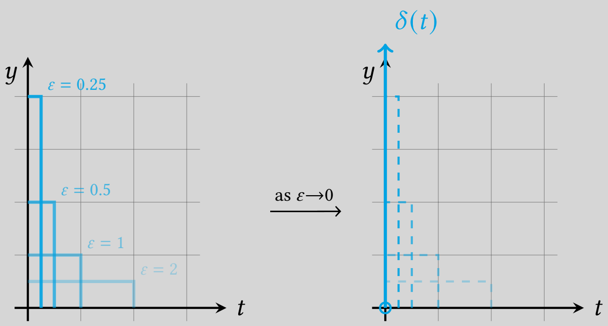
$$\mathcal{L}[u(t-c)] = \frac{e^{-ct}}{s}.$$

- Consider the function

$$f(t) = \begin{cases} 1/\varepsilon, & 0 < t < \varepsilon, \\ 0, & t > \varepsilon. \end{cases}$$

The **Dirac delta function** (or the **unit impulse function**) is defined by taking the limit of $f(t)$ as $\varepsilon \to 0$:

$$\delta(t) = \lim_{\varepsilon \to 0} f(t).$$



Its Laplace transform is given by

$$\mathcal{L}[\delta(t-c)] = e^{-cs}.$$

Theorem 37 (The Second Shifting Theorem). *Let* $F(s) = \mathcal{L}[f(t)]$. *Then,*

$$\mathcal{L}[f(t-c)\,u(t-c)] = e^{-sc}F(s).$$

Equivalently, we may write

$$\mathcal{L}[f(t)\,u(t-c)] = e^{-sc}\mathcal{L}[f(t+c)].$$

Example 38. Describe the graph of

$$y(t) = u(t-a) - u(t-b), \text{ where } 0 \le a < b.$$

Hence, describe the graph of

$$z(t) = f(t)\left[u(t-a) - u(t-b)\right], \text{ where } 0 \le a < b.$$

Solution. By the definition of the unit step function,

$$y(t) = \begin{cases} 0 - 0 = 0, & t < a, \\ 1 - 0 = 1, & a < t < b, \\ 1 - 1 = 0, & t > b. \end{cases}$$

This is simply a function that has value 1 over the interval (a, b) and is zero elsewhere. Thus, the function $z(t)$ takes on the graph of $f(t)$ over the interval (a, b) and zero elsewhere:

$$y(t) = \begin{cases} 0, & t < a, \\ f(t), & a < t < b, \\ 0, & t > b. \end{cases}$$

Example 39. Consider the piecewise function given by

$$f(t) = \begin{cases} 2, & 0 < t < 1, \\ t^2/2, & 1 < t < \pi/2, \\ \cos t, & t > \pi/2. \end{cases}$$

Write $f(t)$ in terms of the unit step function, and hence find the Laplace transform of $f(t)$.

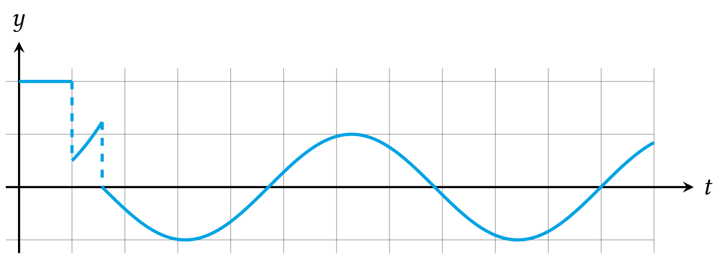


Figure 1: Graph of the function $f(t)$.

Solution. We can rewrite $f(t)$ by taking the products of the function's values with the corresponding difference of unit step functions:

$$\begin{aligned} f(t) &= 2\left(1 - u(t-1)\right) + \frac{t^2}{2}\left[u(t-1) - u\left(t - \frac{\pi}{2}\right)\right] + \cos t \cdot u\left(t - \frac{\pi}{2}\right) \\ &= 2\left(1 - u(t-1)\right) + \frac{t^2}{2} \cdot u(t-1) - \frac{t^2}{2} \cdot u\left(t - \frac{\pi}{2}\right) + \cos t \cdot u\left(t - \frac{\pi}{2}\right). \end{aligned}$$

We now take the Laplace transforms of each term. The first term is fairly straightforward:

$$\mathcal{L}\left[2\left(1 - u(t-1)\right)\right] = 2\left(\frac{1}{s} - \frac{e^{-s}}{s}\right).$$

For the second term, we apply the Second Shifting Theorem,

$$\begin{aligned}\mathcal{L}\left[\frac{t^2}{2}\cdot u(t-1)\right] &= \frac{1}{2}e^{-s}\mathcal{L}\left[(t+1)^2\right]\\ &= \frac{e^{-s}}{2}\mathcal{L}\left[t^2+2t+1\right]\\ &= \frac{e^{-s}}{2}\left(\frac{2!}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right).\end{aligned}$$

Similarly for the third term,

$$\begin{aligned}\mathcal{L}\left[\frac{t^2}{2}\cdot u\left(t-\frac{\pi}{2}\right)\right] &= \frac{1}{2}e^{-s\pi/2}\mathcal{L}\left[\left(t+\frac{\pi}{2}\right)^2\right]\\ &= \frac{1}{2}e^{-s\pi/2}\mathcal{L}\left[t^2+\pi t+\frac{\pi^2}{4}\right]\\ &= \frac{1}{2}e^{-s\pi/2}\left(\frac{2!}{s^3}+\frac{\pi}{s^2}+\frac{\pi^2}{4s}\right).\end{aligned}$$

Finally, we have

$$\begin{aligned}\mathcal{L}\left[\cos t\cdot u\left(t-\frac{\pi}{2}\right)\right] &= e^{-s\pi/2}\mathcal{L}\left[\cos\left(t+\frac{\pi}{2}\right)\right]\\ &= e^{-s\pi/2}\mathcal{L}\left[-\sin t\right]\\ &= -\frac{e^{-s\pi/2}}{s^2+1}.\end{aligned}$$

We can now combine all our results to form $F(s) = \mathcal{L}[f(t)]$:

$$F(s) = 2\left(\frac{1-e^{-s}}{s}\right)+e^{-s}\left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)+e^{-s\pi/2}\left(\frac{1}{s^3}+\frac{\pi}{2s^2}+\frac{\pi^2}{8s}-\frac{1}{s^2+1}\right).$$

Example 40. Find the inverse Laplace transform of

$$F(s) = \frac{1+e^{-\pi s}}{s^2+1}.$$

Solution. The exponential term suggests that $f(t) = \mathcal{L}^{-1}[F(s)]$ is a piecewise function. Applying the Second Shifting Theorem,

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1+e^{-\pi s}}{s^2+1}\right] &= \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]+\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+1}\right]\\ &= \sin t + u(t-\pi)\sin(t-\pi).\end{aligned}$$

Note that since $\sin(t-\pi) = -\sin t$, we have

$$f(t) = \sin t\,(1-u(t-\pi)) = \begin{cases}\sin t, & 0 < t < \pi,\\ 0, & t > \pi.\end{cases}.$$

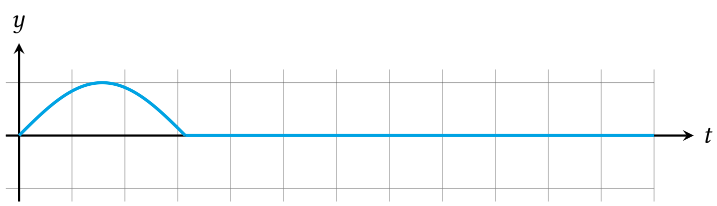


Figure 2: Graph of the function $f(t)$.

Example 41. Consider a damped oscillator which is struck by a hammer at time $t = 1$. The equation of motion is given by the differential equation

$$\ddot{y} + 3\dot{y} + 2y = \delta\left(t - 1\right), \quad y\left(0\right) = \dot{y}\left(0\right) = 0.$$

Determine the function $y\left(t\right)$ that describes the motion of the oscillator.

Solution. Take the Laplace transform of both sides, we have

$$s^2\mathcal{L}\left[y\right] + 3s\mathcal{L}\left[y\right] + 2\mathcal{L}\left[y\right] = e^{-s} \quad \Longrightarrow \quad \mathcal{L}\left[y\right] = \frac{e^{-s}}{s^2 + 3s + 2}.$$

Observe that we may rewrite the above expression—taking a partial fraction decomposition

$$\frac{e^{-s}}{s^2 + 3s + 2} = \frac{e^{-s}}{\left(s+1\right)\left(s+2\right)} = \left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-s}.$$

Taking the inverse of the above function,

$$\mathcal{L}^{-1}\left[\left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-s}\right] = u\left(t-1\right)\left(e^{-(t-1)} - e^{-2(t-1)}\right).$$

Thus, the solution to the initial value problem is given by

$$y\left(t\right) = \begin{cases} 0, & 0 < t < 1, \\ e^{-(t-1)} - e^{-2(t-1)}, & t > 1. \end{cases}$$

5
Partial Differential Equations

5.1 An Introduction

- A **partial differential equation** is an equation involving one or more partial derivatives of a function that depends on two or more variables.
- The **order** of a partial differential equation is the order of the highest derivative in the equation.
- A **solution** of a partial differential equation is a function that satisfies the equation. In general, there are many solutions to a single PDE, and unlike ordinary differential equations, these solutions may all be entirely different from each other
- A partial differential equation is **linear** if it is of the first degree in the unknown function and its derivatives.
- If each of the terms in the linear equation contains either or one of its partial derivatives, then the equation is **homogeneous**; otherwise, it is **nonhomogeneous**.
- In some cases, we can 'construct' a solution to a partial differential equation by the **method of separation of variables**:
  1. Suppose that a solution exists of the form $u(x, y) = X(x)\,Y(y)$.
  2. Replace all expressions of u (e.g., $u, u_x, u_y, \ldots$) in the differential equation with their corresponding expression in terms of X and Y (e.g., $u = XY$, $u_x = X'Y$, $u_y = XY'$, $u_{xy} = X'Y', \ldots$).
  3. Separate all the terms involving x and the terms involving y, and equate both sides to a *separation constant* k.
  4. Solve the ordinary differential equations in x and y to obtain $u(x, y) = X(x)\,Y(y)$.

Theorem 42 (Superposition Principle). *Let $u_1(x, y)$ and $u_2(x, y)$ be solutions of a homogeneous linear partial differential equation. Then, a solution is also given by*

$$u(x, y) = c_1 u_1(x, y) + c_2 u_2(x, y), \text{ for any } c_1, c_2 \in \mathbb{R}.$$

Example 43. The two-dimensional Laplace equation is given by

$$u_{xx} + u_{yy} = 0.$$

Show that the functions $u(x, y) = x^2 - y^2$ and $u(x, y) = e^x \cos y$ are both solutions to the Laplace equation. Hence, conclude that a solution to the Laplace equation is also given by

$$u(x, y) = 5x^2 - 5\left(e^x \cos y + y^2\right).$$

Solution. We compute the partial derivatives of $u(x, y) = x^2 - y^2$:

$$u_{xx} = (2x)_x = 2, \quad u_{yy} = (-2y)_y = -2.$$

Plugging this into the Laplace equation, we find that $u(x, y)$ is indeed a solution:

$$u_{xx} + u_{yy} = 2 - 2 = 0.$$

Similarly, we compute the partial derivatives of $u(x, y) = e^x \cos y$:

$$u_{xx} = (e^x \cos y)_x = e^x \cos y, \quad u_{yy} = (-e^x \sin y)_y = -e^x \cos y,$$

and we verify that $u(x, y)$ satisfies the differential equation:

$$u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0.$$

The Laplace equation is homogeneous and linear: by the Superposition Principle, we find that

$$u(x, y) = 5x^2 - 5\left(e^x \cos y + y^2\right) = 5\left(x^2 - y^2\right) - 5e^x \cos y$$

must also be a solution to the partial differential equation.

Example 44. Use the method of separation of variables to find a solution to

$$u_x + xu_y = 0.$$

Solution. Suppose that a solution of the form $u(x, y) = X(x)\, Y(y)$ exists. Then,

$$u_x(x, y) = X'(x)\, Y(y), \quad u_y(x, y) = X(x)\, Y'(y).$$

Plugging this into the differential equation, we find that

$$X'Y + xXY' = 0.$$

We now perform a separation of variables, noting that the functions X and X' only involve the variable x, and Y, Y'only involve the variable y:

$$X'Y = -xXY' \quad \Longrightarrow \quad \frac{1}{xX} \cdot \frac{dX}{dx} = -\frac{1}{Y} \cdot \frac{dY}{dy}.$$

Observe that equality holds for all values of x and y—this is only possible if both expressions were constant. Thus, we may set

$$\frac{1}{xX} \cdot \frac{dX}{dx} = -\frac{1}{Y} \cdot \frac{dY}{dy} = k \text{ for some } k \in \mathbb{R}.$$

This gives rise to two ordinary differential equations, one in x, and another in y. We solve these equations in turn.

$$\frac{1}{xX}\cdot\frac{dX}{dx} = k \implies \int \frac{1}{X}\,dX = \int kx\,dx \implies \ln|X| = k\frac{x^2}{2} + c,$$

$$-\frac{1}{Y}\cdot\frac{dY}{dy} = k \implies \int \frac{1}{Y}\,dY = \int -k\,dy \implies \ln|Y| = -ky + d.$$

Exponentiating both expressions, we have

$$X(x) = e^{kx^2/2+c} = Ae^{kx^2/2}, \quad Y(y) = e^{-ky+d} = Be^{-ky}.$$

By our initial assumption, the product of $X(x)$ and $Y(y)$ forms the solution to the original differential equation:

$$u(x,y) = X(x)\,Y(y) = \left(Ae^{kx^2/2}Be^{-ky}\right) = Ce^{k\left(x^2/2-y\right)}.$$

Example 45. Use the method of separation of variables to find the solution to the partial differential equation

$$xu_x = u + yu_y$$

that satisfies the conditions $u(1,1) = 2$ and $u(1,2) = 8$.

Solution. Let $u = X(x)\,Y(y)$. Then, $u_x = X'Y$ and $u_y = XY'$, and the differential equation becomes

$$xX'Y = XY + yXY' \implies \frac{x}{X}X' - 1 = \frac{y}{Y}Y'.$$

Setting both sides equal to k, we arrive at two ordinary differential equations which we can solve:

$$\frac{x}{X}X' - 1 = k \implies \int \frac{1}{X}\,dX = \int (k+1)\,\frac{1}{x}\,dx \implies \ln|X| = (k+1)\ln|x| + c,$$

$$\frac{y}{Y}Y' = k \implies \int \frac{1}{Y}\,dY = \int k\frac{1}{y}\,dy \implies \ln|Y| = k\ln|y| + d.$$

Exponentiating both sides, we thus find that

$$X(x) = Ax^{k+1}, \quad Y(y) = By^k.$$

So, a solution to the partial differential equation is given by

$$u(x,y) = X(x)\,Y(y) = Cx^{k+1}y^k, \text{ where } C \in \mathbb{R}.$$

To find a particular solution, we use the given conditions.

$$\begin{aligned} u(1,1) &= C\cdot 1\cdot 1 = 2 &\implies& \quad C = 2, \\ u(1,2) &= 2\cdot 1\cdot 2^k = 8 &\implies& \quad k = 2. \end{aligned}$$

Thus, the particular solution is given by

$$u(x,y) = 2x^3y^2.$$

5.2 *The Heat Equation*

► The dispersion of heat on a metal rod of length ℓ is described by the **heat equation**

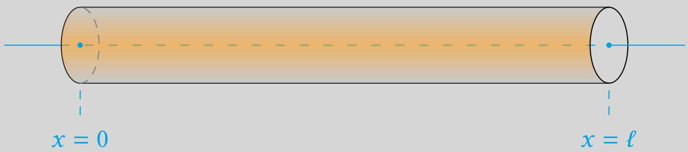
$$u_t = c^2 u_{xx}, \quad 0 < x < \ell, \quad t > 0.$$

The constant c^2 is known as the *thermal diffusivity* of the metal, and the solution $u(x,t)$ describes the temperature of the rod at a given point x and time t. Assuming that the rod is insulated at its ends $x = 0$ and $x = \ell$, we have the *boundary conditions*

$$u(0,t) = 0, \qquad u(\ell,t) = 0.$$

If the initial distribution of heat is given by the function $f(x)$, then we have the *initial condition*

$$u(x,0) = f(x).$$



Example 46. Use the method of separation of variables to find a solution to the heat equation given the initial and boundary conditions

$$u_t = c^2 u_{xx}, \quad u(0,t) = u(\ell,t) = 0, \quad u(x,0) = f(x).$$

Solution. Let $u(x,t) = X(x)T(t)$. Then, $u_t = XT'$, $u_{xx} = X''T$: plugging this into the heat equation,

$$XT' = c^2 X''T \quad \Longrightarrow \quad \frac{1}{c^2} \cdot \frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}.$$

Setting both sides equal to a constant k, we obtain two ordinary differential equations. The first-order differential equation in t is easy to solve:

$$\int \frac{1}{T}\,dT = \int c^2 k\,dt \quad \Longrightarrow \quad \ln T = c^2kt + d \quad \Longrightarrow \quad T = Ae^{c^2kt}.$$

Now, consider the second-order differential equation in x:

$$\frac{1}{X}\frac{d^2X}{dx^2} = k \quad \Longrightarrow \quad X'' - kX = 0.$$

This is a second-order homogeneous linear differential equation: the characteristic equation is given by

$$\lambda^2 - k = 0 \quad \Longrightarrow \quad \lambda = \pm\sqrt{k}.$$

The nature of the solution $X(x)$ now depends on the value of k, but we know that it will involve either exponentials or sines and cosines.

By our initial conditions, $u(x,t)$ must be 0 at both $x = 0$ and $x = \ell$. This would be impossible of $u(x,t)$ only involved exponential terms; hence, $X(x)$ must be trigonometric. In particular, we require that $k < 0$, which yields the solution

$$X(x) = c_1 \cos\left(\sqrt{-k}x\right) + c_2 \sin\left(\sqrt{-k}x\right).$$

Thus, a general solution to the heat equation is given by

$$u(x,t) = X(x)\,T(t) = e^{c^2kt}\left(\alpha \cos\left(\sqrt{-k}x\right) + \beta \sin\left(\sqrt{-k}x\right)\right).$$

We now find the values of the constants α, β, and k. Given the boundary conditions $u(0,t) = 0$ and $u(t,\ell) = 0$,

$$u(0,t) = e^{c^2kt}(\alpha \cos 0 + \beta \sin 0) = \alpha e^{c^2kt} = 0 \quad \Longrightarrow \quad \alpha = 0,$$

$$u(\ell,t) = e^{c^2kt}\left(\beta \sin\left(\ell\sqrt{-k}\right)\right) = 0 \quad \Longrightarrow \quad \beta \sin\left(\ell\sqrt{-k}\right) = 0.$$

Since α and β cannot both be 0, the second equation implies that

$$\sin\left(\ell\sqrt{-k}\right) = 0 \quad \Longrightarrow \quad \sqrt{-k} = \frac{n\pi}{\ell}, \text{ for } n \in 0, 1, 2, \ldots.$$

Thus, $k = -n^2\pi^2/\ell^2$. Thus, our solution is now given by

$$u(x,t) = \beta_n e^{-c^2n^2\pi^2t/\ell^2} \sin\left(\frac{n\pi}{\ell}x\right).$$

In order to deduce β_n, we use the initial condition $u(0,x) = f(x)$:

$$u(x,0) = \beta_n \sin\left(\frac{n\pi}{\ell}x\right) = f(x).$$

Observe that for different functions $f(x)$, the heat equation will have different coefficients and, as expected, different particular solutions.

Example 47. Solve the partial differential equation

$$u_t = 2u_{xx}, \qquad 0 < x < 3, \qquad t > 0,$$

given the boundary conditions $u(0,t) = u(3,t) = 0$ and the initial condition $u(x,0) = 5\sin(4\pi x)$.

Solution. This is simply the heat equation with $c^2 = 2$, $\ell = 3$, and $f(x) = 5\sin(4\pi x)$. We have already found its solution:

$$u(x,t) = \beta_n e^{-2n^2\pi^2t/9} \sin\left(\frac{n\pi}{3}x\right),$$

and it remains to determine the values of n and β_n. By the initial condition $u(x,0) = 5\sin(4\pi x)$, we have at $t = 0$

$$\beta_n \sin\left(\frac{n\pi}{3}x\right) = 5\sin(4\pi x).$$

Observe that n must hence be 12, and so $\beta_{12} = 5$. Thus, a solution to the heat equation under these conditions is given by

$$u(x,t) = 5e^{-2\cdot 144\cdot\pi^2t/9} \sin(4\pi x) = 5e^{-32\pi^2t} \sin(4\pi x).$$

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n, for $n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$

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